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Graceful Labeling of Arbitrary Supersubdivision of Grid graph and Cyclic snake

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Abstract— Aim of this paper is to prove the graph obtained by arbitrary supersubdivision of grid graph $P_s \times P_t$ is graceful and the arbitrary supersubdivision of k-block generalized cyclic snake with string-1 is graceful. Also we define a k- block generalized cyclic snake in the increasing length and we prove that such graph is graceful.

Index Terms— Arbitrary supersubdivision, Generalized cyclic snake, Graceful labeling, Grid graph.

1 INTRODUCTION

We begin with simple, finite, undirected and connected graph $G(p, q)$. A *graceful labeling* of G is an injection from the set of its p vertices to the set $\{0, 1, 2, ..., q\}$ such that the values of the edges are all integers from 1 to q, the value of an edge being the absolute value of the difference between the integers attributed to its end vertices.

Sethuraman and Selvaraju [1] have introduced a new method of construction called supersubdivision of a graph and showed that arbitrary supersubdivisions of paths are graceful. They conjectured that paths and stars are the only graphs for which every supersubdivision is graceful. Barrientos [2] disproved this conjecture by proving that every supersubdivision of a y-tree is graceful. Sethuraman and Selvaraju [1] proved that every connected graph has some supersubdivision that is graceful. They pose that question as to whether some supersubdivision is valid for disconnected graphs [3]. After that Sekar and Ramachandran [4] proved that arbitrary Supersubdivision of disconnected graph is graceful.

The planar grids $P_m \times P_n$ are graceful was proved by Acharya and Gill [5] in 1978. Jungreis and Reid [6] showed that the grids $P_m \times P_n$ are harmonious when $(m,n) \neq (2,2)$. Vaidya, Dani, Vihol and Kanani [7] proved that an arbitrary supersubdivision of grid graph $P_s \times P_t$ is strongly multiplicative and they pose that question as to whether parallel result can be investigated corresponding to other graph labeling techniques [3]. Shiu and Kwong [8] determine the friendly index of the grids $P_2 \times P_n$.

A kC_n *-snake* is a connected graph with k blocks, each of the blocks is isomorphic to the cycle C_n, such that the block-cut-vertex graph is a path. Following Chartrand, Lesniak[9], by a block-cut-vertex graph of a graph G we mean the graph whose vertices are the blocks and cut-vertices of G where two vertices are adjacent if and only if one vertex is a block and the other is a cut-vertex belonging to the block. This graph was first introduced by Barrientos[2] and he proves that kC_4-snakes are graceful and later it was discussed by Badr[10] as generalization of the concept of triangular snake introduced by Rosa[11]. Also Badr [10] proved that kC_4-snake, linear kC_n-snake, even kC_8-snake and even kC_{12}-snakes are odd graceful. Lourdusamy and Seenivasan [12] proved that kC_n-snakes are means graphs and every cycle has a supersubdivision that is a mean graph. They defined a generalized kC_n-snake in the same way as a C_n-snake except that the sizes of the cycle blocks can vary. They also proved that generalized kC_n-snakes are mean graphs. For detail survey on graph labeling in the field of arbitrary supersubdivision one can refer to Gallian [3], Kathiresan and Amutha [13-14]. We call a kC_n-snake as a k- block cyclic snake and a generalized kC_n-snake as a k- block generalized cyclic snake.

A kC_n *-snake* contains $M = nk$ edges and $N = (n-1)k+1$ vertices. Among these vertices, $k - 1$ vertices have degree 4 and the other vertices of degree 2. Let $w_1, w_2, ... w_{k-1}$ be the consecutive cut-vertices of G. Let d_i be the distance between w_i and w_{i+1} in G for $1 \leq i \leq k-2$, the string $(d_1, d_2, ... d_{k-2})$ of integers characterizes the graph G in the class of n-cyclic snakes.

For example, refer Figure-1 for a $2C_4$-snake with a cut vertex. Now we can construct two different $3C_4$ *-snake* from a $2C_4$ *-snake*, the first is with string- 1 (Figure-2) and the second is with string- 2 (Figure-3).

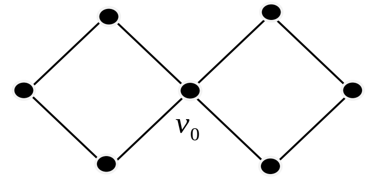

Figure-1: $2C_4$-snake with a cut vertex

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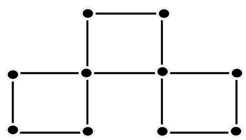

Figure-2: $3C_4$ **-snake with string- 1**

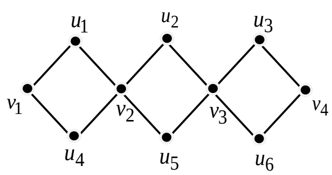

Figure-3: $3C_4$ **-snake with string- 2**

In this paper we prove the following results for graceful labeling.

1. The graph obtained by arbitrary supersubdivision of grid graph $P_s \times P_t$ is graceful.
2. An arbitrary supersubdivision of generalized kC_n-snake with string-1 is graceful.
3. The generalized kC_n-snake in the increasing odd length with string-1 is graceful.
4. An arbitrary supersubdivision of triangular snake with length k is graceful.

Definition 1.1

Let G be a graph with q edges. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G. A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily).

Definition 1.2

Let $G(p, q)$ be a g*eneralized* kC_n *-snake* with string-1 and $(n_1, n_2, \ldots, n_k)$ be the string length of cycles of G, then G contains contains $q = \sum_{l=1}^{k} n_l$ edges and $p = \sum_{l=1}^{k} n_l - (k-1)$ vertices. Name the vertices of G as $V_1, V_2, \ldots, V_p$ as shown in Figure-4, we observe that there is a shortest path $V_1, V_{n_1}, V_{n_1+n_2-1}, V_{n_1+n_2+n_3-2}, \ldots, V_p$ along the cut vertices of G, we can call it as *the Cut vertex path of G* and there is a *Hamiltonian path* $V_1, V_2, \ldots, V_p$ as shown in Figure-4 which covers all the vertices of G. It is clear that the Cut vertex path and Hamiltonian path are distinct in G and the union of Cut vertex path and Hamiltonian path covers all the edges of G.

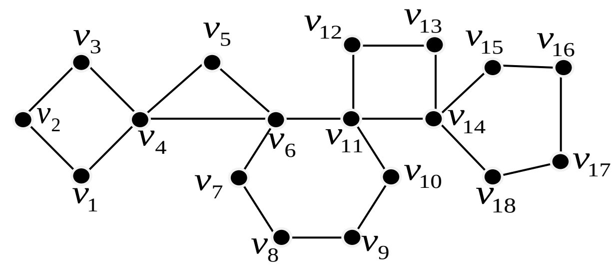

Figure-4: Generalized $5C_n$ **-snake with string length** $(n_1, n_2, n_3, n_4, n_5) = (4, 3, 6, 4, 5)$

Definition 1.3

If $(n_1, n_2, \ldots, n_k) = (3, 4, \ldots, k+2)$ is the string length of a generalized kC_n-snake then it is said to be a generalized kC_n-snake in the increasing length.

If $(n_1, n_2, \ldots, n_k) = (3, 5, \ldots, 2k+1)$ is the string length of a generalized kC_n-snake then it is said to be a generalized kC_n-snake in the increasing odd length.

2 Main Results

In this paper we prove our main results that the graph obtained by arbitrary supersubdivision of grid graph $P_s \times P_t$ is graceful and the arbitrary supersubdivision of *k*-block generalized cyclic snake with string-1 is graceful.

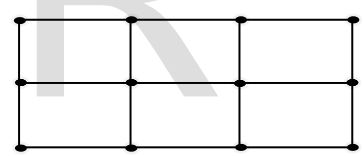

Figure-5: Grid graph $P_4 \times P_3$

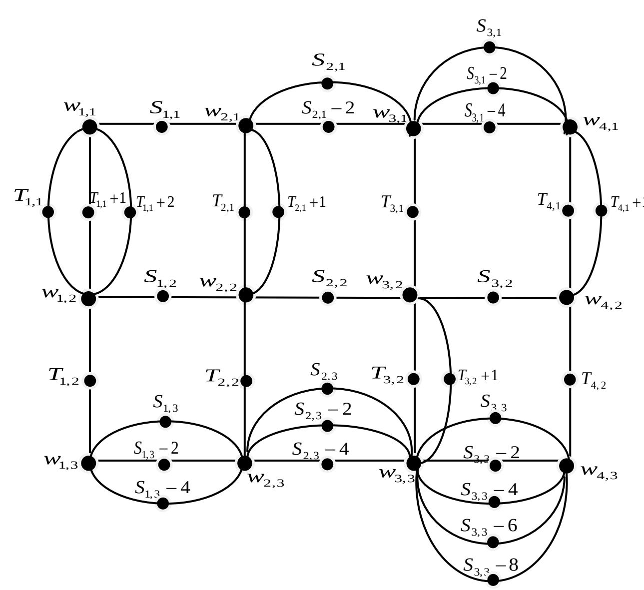

Figure-6: Some Arbitrary Supersubdivision of $P_4 \times P_3$

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Theorem 2.1: The arbitrary supersubdivision of grid graph is graceful.

Proof

Let $G(p,q)$ be a grid graph $P_s \times P_t$. Name the vertices of G by $w_{i,j}$, $1 \le i \le s$ & $1 \le j \le t$. Observe that there are t times of P_s paths say $P_s^{(j)}$, $1 \le j \le t$ and s times of P_t paths say $P_t^{(i)}$, $1 \le i \le s$. Let $P_s^{(j)}$ be a path with successive vertices $w_{1,j}, w_{2,j}, \ldots, w_{s,j}$ for $1 \le j \le t$ and let $P_t^{(i)}$ be a path with successive vertices $w_{i,1}, w_{i,2}, \ldots, w_{i,t}$ for $1 \le i \le s$. Let $e_{i,j}$ be the edges of the paths $P_s^{(j)}$ having end points $w_{i,j}$ & $w_{i+1,j}$ for $1 \le i \le s-1$ & $1 \le j \le t$ and let $e'_{i,j}$ be the edges of the paths $P_t^{(i)}$ having end points $w_{i,j}$ & $w_{i,j+1}$ for $1 \le i \le s$ & $1 \le j \le t-1$.

Let H be an arbitrary supersubdivision of G, that is every edge $e_{i,j}$ is replaced by a complete bipartite graph $K_{2,m_{i,j}}$, where $m_{i,j}$ is any positive integer and every edge $e'_{i,j}$ is replaced by a complete bipartite graph $K_{2,n_{i,j}}$, where $n_{i,j}$ is any positive integer. Note that the two vertices of the 2-vertices part of $K_{2,m_{i,j}}$ get the labels $w_{i,j}$ & $w_{i+1,j}$ and $S_{i,j}, S_{i,j}-2, S_{i,j}-4, \ldots, S_{i,j}-2(m_{i,j}-1)$ are assigned to the $m_{i,j}$ vertices of $m_{i,j}$-vertices part of $K_{2,m_{i,j}}$ [Refer Figure-6 and Figure-7]. Also the two vertices of the 2-vertices part of $K_{2,n_{i,j}}$ get the labels $w_{i,j}$ & $w_{i,j+1}$ and $T_{i,j}, T_{i,j}+1, \ldots, T_{i,j}+n_{i,j}-1$ are assigned to the $n_{i,j}$ vertices of $n_{i,j}$-vertices part of $K_{2,n_{i,j}}$ [Refer Figure-6 and Figure-7]. Let N be the number of vertices and M be the number of edges of H, where N and M are defined below.

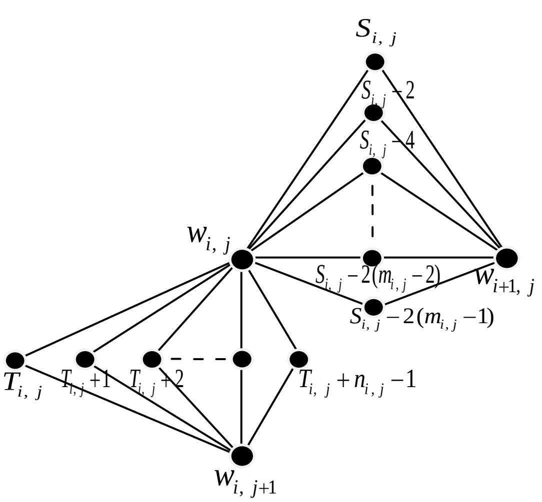

Figure-7

Let

- $M_j = \sum_{i=1}^{s-1} m_{i,j}$, $1 \le j \le t$
- $R_j = \sum_{i=1}^{s} n_{i,j}$, $1 \le j \le t-1$
- $E_1 = M = 2\left[\sum_{l=1}^{t} M_l + \sum_{l=1}^{t-1} R_l\right]$
- $N = st + \dfrac{M}{2}$
- $E_j = E_{j-1} - 2M_{j-1} - R_{j-1}$, $2 \le j \le t$

Define

$$w_{i,j} = (i-1) + \sum_{l=1}^{j-1} R_l, \; 1 \le i \le s, \; 1 \le j \le t$$

$$S_{i,j} = E_j - 2\sum_{l=1}^{i-1} m_{l,j} + i - 1, \; 1 \le i \le s-1, \; 1 \le j \le t$$

$$T_{i,j} = E_{j+1} + \sum_{l=1}^{i-1} n_{l,j} + i, \; 1 \le i \le s, \; 1 \le j \le t-1$$

It is clear from the above labeling that H has an injection from the set of its N vertices to the set $\{0, 1, 2, \ldots, M\}$ and the M edges of H have distinct labels from 1 to M. So H is graceful.

Example 2.2: By using Theorem 2.1, we can give graceful labeling for the grid graph which is in Figure-6. (Refer Figure-8 for the graceful labeling of the graph which is in Figure-6). It is clear that the graph has an injection from the set of its $N(=45)$ vertices to the set $\{0, 1, 2, \ldots, M(=66)\}$ and the $M(=66)$ edges of that graph have distinct labels from 1 to $M(=66)$, so the graph is graceful.

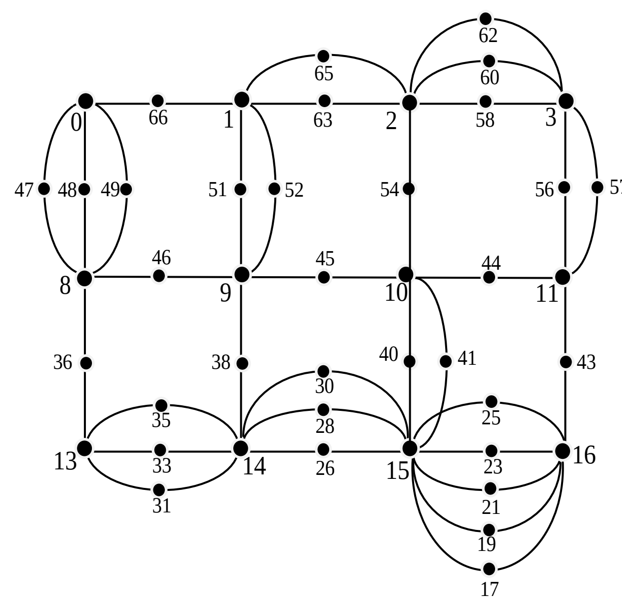

Figure-8

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Theorem 2.3: From $n \geq 3$, there exist an arbitrary supersubdivision of *k*-block generalized cyclic snake with string-1 that is graceful for $k \geq 2$.

Proof

Let $G(p, q)$ be the *k*- block generalized cyclic snake with string-1 and $(n_1, n_2, \ldots, n_k)$ be the string length of cycles of *G*, then *G* contains $q = \sum_{l=1}^{k} n_l$ edges and $p = \sum_{l=1}^{k} n_l - (k-1)$ vertices. Name the vertices of *G* as $V_1, V_2, \ldots, V_p$ as shown in Figure-4. Let P'_{k+1} be the cut vertex path $V_1, V_{n_1}, V_{n_1+n_2-1}, V_{n_1+n_2+n_3-2}, \ldots, V_p$ with length *k*+1 along the cut vertices of *G*, we may name them as $V'_1, V'_2, \ldots, V'_{k+1}$ respectively and let P_p be the Hamiltonian path $V_1, V_2, \ldots, V_p$ of *G* with length *p*. Let $e_i = v_i v_{i+1}, 1 \leq i \leq p-1$ denote the edges of the path P_p for $1 \leq i \leq p-1$ and let $e'_j = v'_j v'_{j+1}, 1 \leq j \leq k$ denote the edges of the path P'_{k+1}.

Let *H* be an arbitrary supersubdivision of *G*, that is every edge $e_i, 1 \leq i \leq p-1$ of the path P_p is replaced by a complete bipartite graph K_{2,m_i} where m_i is any positive integer and every edge $e'_j, 1 \leq j \leq k$ of the path P'_{k+1} is replaced by a complete bipartite graph K_{2,m'_j} where $m'_j = n_j - 1, 1 \leq j \leq k$ is any positive integer. Note that the two vertices of the 2-vertices part of K_{2,m_i} get the labels $v_i \& v_{i+1}$ and $N_i, N_i - 2, N_i - 4, \ldots, N_i - 2(m_i - 1)$ are assigned to the m_i vertices of m_i-vertices part of K_{2,m_i} as in Figure-9. Also the two vertices of the 2-vertices part of K_{2,m'_j} get the labels $v'_j \& v'_{j+1}$ and $N'_j, N'_j - 1, N'_j - 2, \ldots, N'_j - n_j - 2$ are assigned to the m'_j vertices of m'_j-vertices part of K_{2,m'_j} as in Figure-10. Let *N* be the number of vertices and *M* be the number of edges of *H*, where *N* and *M* are defined below.

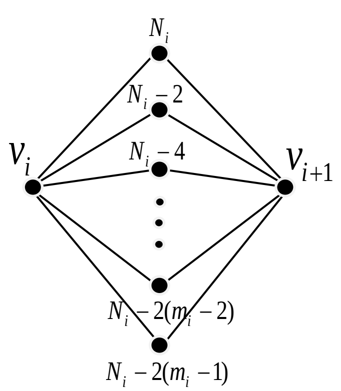

Figure-9

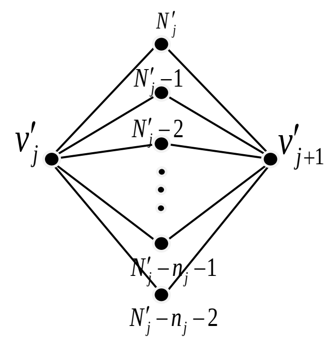

Figure-10

Define

$$M = 2\left\lfloor \sum_{i=1}^{N-1} m_i + \sum_{j=1}^{k} m'_j \right\rfloor$$

$$N = p + \frac{M}{2}$$

$$V_i = i - 1,\ 1 \leq i \leq N \text{ and } V'_1 = 0$$

$$V'_j = \left\lfloor \sum_{l=1}^{j-1} n_l - (j-2) \right\rfloor - 1,\ 2 \leq j \leq k+1$$

$$N_i = M - 2\sum_{l=1}^{i-1} m_l + (i-2),\ \ 1 \leq i \leq N-1$$

$$N'_j = 2(N-1) - \sum_{l=1}^{j-1} m'_l,\ \ 1 \leq j \leq k$$

It is clear from the above labeling that *H* has an injection from the set of its *N* vertices to the set $\{0, 1, 2, \ldots, M\}$ and the *M* edges of *H* have distinct labels from 1 to *M*. So *H* is graceful.

Example 2.4: Refer Figure-11 for the graceful labeling of some arbitrary supersubdivision of the graph which is in Figure-4.

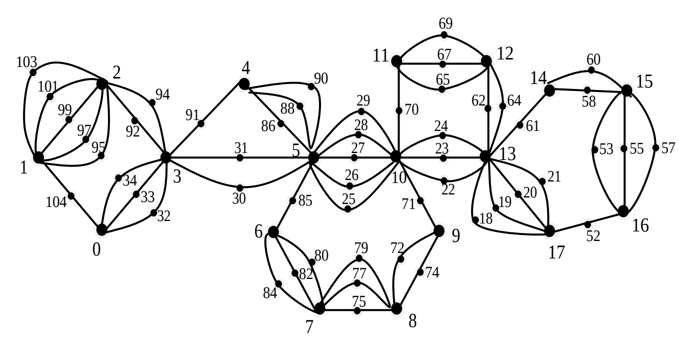

Figure-11

Corollary 2.5: From $n \geq 3$, there exist an arbitrary supersubdivision of kC_n-snake with string-1 that is graceful for $k \geq 2$.

Proof:

Let $G(p, q)$ be the kC_n-snake with string-1 then *G* contains $q = nk$ edges and $p = (n-1)k + 1$ vertices. Let *H* be an arbitrary supersubdivision of *G*. By replacing the string length

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values $n_1 = n_2 = \ldots = n_k = n$, the generalized kC_n-snake with string-1 will become a kC_n-snake with string-1. So by Theorem 2.3, it is clear that from $n \geq 3$, there exist an arbitrary supersubdivision of kC_n-snake with string-1 that is graceful for $k \geq 2$.

Note: Similarly we can prove an arbitrary supersubdivision of *k*-block generalized cyclic snake in the increasing length and also in the increasing odd length with string-1 are graceful for $k \geq 2$.

Corollary 2.6: An arbitrary supersubdivision of triangular snake with length *k* is graceful for $k \geq 2$.

Proof:

Let $G(p, q)$ be the triangular snake with length *k*, then *G* contains $q = 3k$ edges and $p = 2k + 1$ vertices and let *H* be an arbitrary supersubdivision of *G*. By replacing the string length values $n_1 = n_2 = \ldots = n_k = 3$, the *k*- block generalized cyclic snake with string-1 will become triangular snake with length *k*. So by Theorem 2.3, it is clear that for $k \geq 2$, an arbitrary supersubdivision of triangular snake with length *k* is graceful.

Theorem 2.7: A *k*-block generalized cyclic snake in the increasing odd length with string-1 is graceful for $k \geq 2$.

Proof

Let $G(p, q)$ be the *k*-block generalized cyclic snake in the increasing odd length with string-1 and name the vertices of *G* as $V_1, V_2, \ldots, V_p$ as shown in Figure-4, then *G* contains $q = \sum_{l=1}^{k} n_l$ edges and $p = \sum_{l=1}^{k} n_l - (k-1)$ vertices.

Define

$$V_{2i-1} = i - 1, \ 1 \leq i \leq \frac{p+1}{2}$$

$$V_{2i} = q + 1 - i, \ 1 \leq i \leq \frac{p-1}{2}$$

It is clear from the above labeling that *G* has an injection from the set of its *p* vertices to the set $\{0, 1, 2, \ldots, q\}$ and the *q* edges of *G* have distinct labels from 1 to *q*. So *G* is graceful. For example see Figure-12.

Example 2.8: Refer Figure-12 for the graceful labeling of a 4-block generalized cyclic snake in the increasing odd length with string-1.

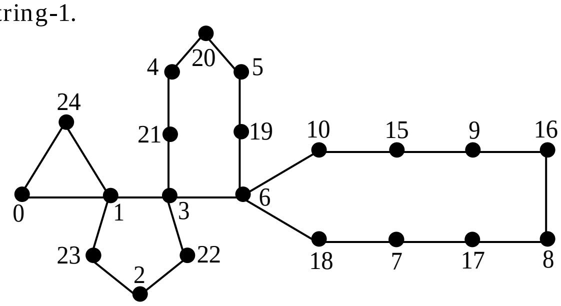

Figure-12

3 Conclusion

In this paper we have shown that the arbitrary supersubdivision of grid graph $P_s \times P_t$ is graceful and there exist an arbitrary supersubdivision of *k*- block generalized cyclic snake with string-1 is graceful for $k \geq 2$. From these results we extended our results that from $n \geq 3$, there exist an arbitrary supersubdivision of kC_n-snake with string-1 that is graceful for $k \geq 2$ and an arbitrary supersubdivision of triangular snake with length *k* is graceful for $k \geq 2$.

4 Remarks

'Supersubdivision of a graph' can be used as a powerful operation to get larger size graphs from a given graph. We observe that, from the arbitrary supersubdivision of a grid graph and as well as from the generalized kC_n-snake, we can get some graphs with infinite length.

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